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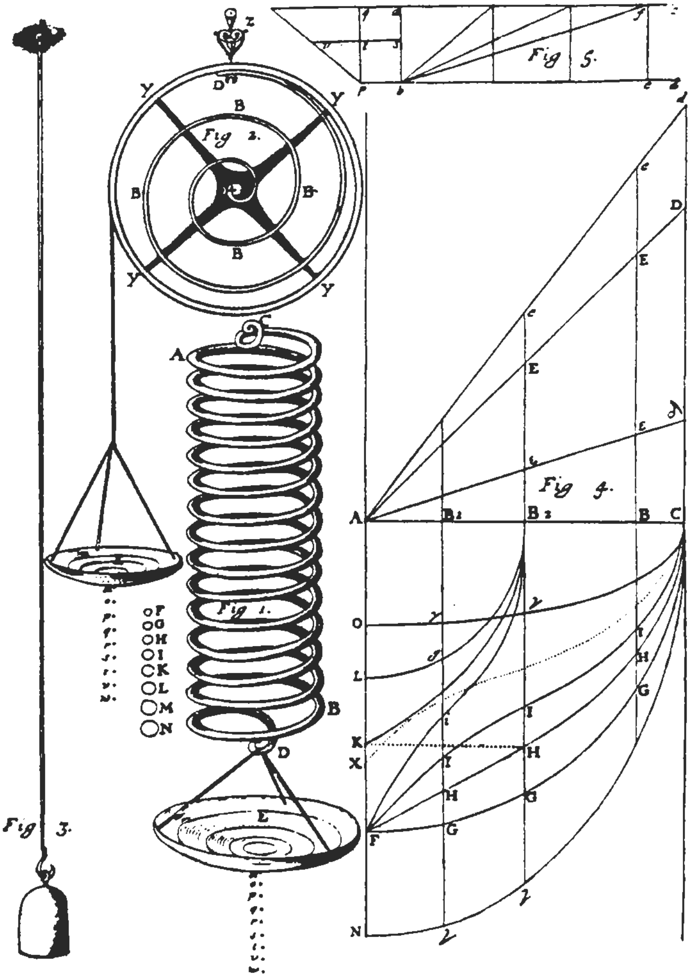

Hooke R., *A description of helioscopes, and some other instruments* (1675)

THE THEORY OF ELASTICITY BETWEEN MOLECULAR AND CONTINUUM APPROACH IN THE XIX CENTURY

Federico Foce[1]

Summary: The incompatibility between the theory of elasticity based on the continuum hypothesis and the results obtained from the atomistic hypothesis has interested many historians. Isn't it another expression of the opposition between the adherents to the idea of force assumed as a primitive element of mechanics and those who, like d'Alembert, Carnot and later Saint-Venant, considered it as a derived quantity definable in strictly kinematical terms? Once more the problem seems to lie in the comprehension of the notion of stress according to the two alternative methodological approaches.

Résumé : L'incompatibilité entre la théorie de l'élasticité basée sur l'hypothèse continuiste et les résultats obtenus dans l'hypothèse atomiste a arrêté de nombreux historiens. Ne serait-elle pas une autre expression de l'opposition entre les adeptes de la force (ou de la tension) considérée comme grandeur fondamentale et ceux qui comme d'Alembert et Carnot, ne veulent voir que l'aspect cinématique de celle-ci, que le mouvement que la force produit? Une fois de plus, l'enjeu semble se situer au niveau de la compréhension de la notion de tension en suivant l'une ou l'autre approche méthodologique.

Introduction

If it is a primary need of the historian to weave the narration with a measured gap from his own material of reflection, then there is no doubt that the celebrated *History of the theory of elasticity* by Todhunter and Pearson[2] constitutes a fundamentally defective text, a great fresco prematurely conceived "in corso d'opera". From Galilei to Lord Kelvin the progresses registred by the *Strength of materials* have certainly been extraordinary, however that history stops too soon to be able to extract from it the

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[2] I. TODHUNTER, K. PEARSON, *A History of the Theory of Elasticity and of the Strength of Materials from Galilei to the present time*, vol. 1, Cambridge, 1886; vol. 2, Cambridge, 1893.

certainty of a finally acquired goal. In reality, the three ponderous volumes usually indicate as the most complete and informed work on the evolution of pre twentieth-century mechanics of solids, are lacking a real and proper epilogue. On second thoughts, this could not be otherwise if one considers that, in spite of a nowadays consolidated historiographical view, at the end of the XIX century the theory of elasticity still awaited a definite word capable of removing, once and for all, the doubts that had troubled its developments between molecular and continuum formulation.

Paris 1823: the contradictory beginning on the *Bulletin de la Société philomatique*

From this point of view, the volume of the *Bulletin de la Société Philomatique de Paris* of the year 1823 represents a singular event in our history, one of those events which would be spontaneous to be assumed as *a quo* term in the explicit definition of the features proper to one or the other formulation of the elasticity of solid bodies. That volume contains in fact the fundamental results obtained by Navier[3] and Cauchy[4] starting from the opposite models of the *assemblage de molécules* and of the *masse continue*.

But it would be difficult to maintain such a thesis. Even if the *Mémoire* by Navier and the *Recherches* by Cauchy undoubtedly represent methodological paradigms perfectly responding to the two alternative conceptions of the material universe, the analysis of the texts and history itself show how much there was still to clarify after that memorable date[5].

It is known that, according to the continuum theory of Cauchy[6], the stress-strain equations $\sigma_{ij} = c_{ijpq}\,\varepsilon_{pq}$ contain 36 different elastic constants c_{ijpq} (with $i, j, p, q = 1, 2, 3$)

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[3] L. NAVIER, *Sur les lois de l'équilibre et du mouvement des corps solides élastiques*, Bulletin des sciences par la Société Philomatique de Paris, 1823, pp. 177-181; abstract of the memoir presented to the Académie des Sciences on May 14, 1821 and published with the title *Mémoire sur les lois de l'équilibre et du mouvement des corps solides élastiques*, Mémoires de l'Institut National, vol. 7, 1827, pp.375-393.

[4] A. CAUCHY, *Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques*, Bulletin des sciences par la Société Philomatique de Paris, 1823, pp. 9-13, presented to the Académie Royale des Sciences on Septembre 30, 1822.

[5] Regarding this theme see F. FOCE, *La teoria molecolare dell'elasticità dalla fondazione ottocentesca ai nuovi sviluppi del XX secolo*, Doctoral thesis, Florence, 1993.

[6] The content of the *Recherches* is made explicit by A. CAUCHY in the memoirs *De la pression ou tension dans un corps solid*, Exercices de Mathématiques, vol. 2, 1827, pp. 42-56; *Sur la condensation et la dilatation des corps solides*, idem, pp. 60-69; *Sur les relations qui existent, dans l'état d'équilibre d'un corps solide ou fluide, entre les pressions ou tensions et les forces accélératrices*, idem, pp. 108-111. In the memoir *Sur les équations qui expriment les conditions d'équilibre ou les lois du mouvement intérieur d'un corps solide, élastique, ou non élastiques*, idem, vol. 3, 1828, pp. 160-187, A. CAUCHY

for the general case of the aelotropic bodies, and two constants, k and K, for the isotropic bodies. On the contrary, the molecular formulation, supported by Poisson in the context of his *Mécanique physique*[7] and parallelly developed by Cauchy himself when he substituted his continuum theory in the late twenties[8], leads to equations containing 15 elastic constants for the aelotropic bodies and only one constant k for the isotropic ones; in this case, in fact, the elastic coefficients satisfy the equalities $c_{ijpq} = c_{ipjq}$, that is $k = K$ for isotropy, which have entered into modern literature with the justified title "Cauchy relations"[9].

It is really singular that these conflicting results were at first hidden by Cauchy's own attempt to compare the two versions of isotropic elasticity[10]. It seemed perhaps spontaneous to think that, by interpreting correctly the formulae of both theories, identical

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introduces for the first time the stress-strain relations for isotropic bodies; he initially writes (p. 171) the uni-constant formulae

$$\sigma_{11} = k\,\varepsilon_{11} \qquad \sigma_{23} = k\,\varepsilon_{23}\,,$$

that he soon abandons (p. 177) in favour of the bi-constant ones

$$\sigma_{11} = k\,\varepsilon_{11} + K(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \qquad \sigma_{23} = k\,\varepsilon_{23}$$

to which he then refers for his continuum theory. Finally, in the memoir *Sur l'équilibre et le mouvement intérieur des corps considérés comme des masses continues*, idem, vol. 4, 1829, pp. 293-319, Cauchy furnishes the stress-strain equations in terms of 36 different elastic constants for the general case of aelotropic bodies.

[7] S. D. POISSON, *Mémoire sur l'équilibre et le mouvement des corps élastiques*, Mémoires de l'Académie Royale de Sciences de l'Institut, vol. 8, 1829, pp. 353-570; *Mémoire sur les équations générales de l'équilibre et du mouvement des corps solides élastiques et des fluides*, Journal de l'École polytechnique, vol. 13, Cahier 20, 1831, pp. 1-174. Regarding the position supported by Poisson in favour to Laplacian physics see the essay by D.H. ARNOLD, *The Mécanique Physique of Simeon Denis Poisson: the evolution and isolation in France of his approach to physical theory (1800-1840)*, Archive for history of exact sciences, vol. 28, 1983; vol. 29, 1984.

[8] A. CAUCHY, *Sur l'équilibre et le mouvement d'un système de points matériels sollicités par des forces d'attraction ou de répulsion mutuelle*, Exercices de Mathématiques, vol. 3, 1828, pp. 188-212; *De la pression ou tension dans un système de points matériels*, idem, pp. 213-236; *Sur les équations differéntielles d'équilibre ou de mouvements pour un système de points matériels sollicités par des forces d'attraction ou de répulsion mutuelle*, idem, vol. 4, 1829, pp. 129-139. Regarding Cauchy's position towards molecular mechanics see the recent essays by B. BELHOSTE, *Augustin-Louis Cauchy. A Biography*, New York, 1991, p. 106 and by A. DAHAN DALMEDICO, *Mathématisations. Augustin-Louis Cauchy et l'École Française*, Paris, 1992, in particular, Chapter XI.

[9] On the problem of the "Cauchy relations" and their modern interpretation see E. BENVENUTO, F. FOCE, *Alle origini della micro-meccanica dei materiali. Cenni storici sul problema delle "relazioni di Cauchy"*, Atti Convegno AIMETA "Meccanica dei materiali e delle strutture" (Amalfi, June 3-5 1991), Napoli, 1992, pp. 7-13; E. BENVENUTO, M. CORRADI, F. FOCE, *Considerazioni critiche sulle cosiddette "relazioni di Cauchy"*, Atti XI Congresso AIMETA (Trento, Sept. 28-Oct. 2, 1992), Trento, 1992, pp. 79-84.

[10] The motivation of this forced comparison is treaceable in the formal analogy between the equations of the continuum theory, containing two distinct elastic constants k and K and written considering the natural state of the body (see the memoir *Sur les équations qui expriment les conditions d'équilibre ou les lois du mouvement intérieur d'un corps solide, élastique ou non élastique*, cit., in particular, p. 177) and the equations of the molecular theory containing a single elastic constant, to which a second constant is added representing the eventual stress that is present when the initial state of the body is not the natural one (see the memoir *Sur l'équilibre et le mouvement d'un système de points matériels sollicités par des forces d'attraction ou de répulsion mutuelle*, Ibidem, in particular p. 205).

results could be obtained even moving from the two different methodological approaches. Incontestable evidence of this initial misunderstanding is treaceble, for instance, in the erroneous conclusions drawn by Wertheim[11] in the attempt to interpret his important experimental results in the light of a new uni-constant version for isotropy. Not even the clarifying intervention of Clausius[12] on the real meaning of the double series of formulae given by Cauchy could avoid Lamé[13] declaring himself in favour of the bi-constant theory without rejecting the original molecular model.

Green's "new method" and the English approach to elasticity

In his modern rewriting of the *continuum mechanics* Truesdell distinguishes two possible methods of understanding the concept of springiness: the first one, attributed to Cauchy, defines a perfectly elastic body as one in which *the stress is a function of the strain*; the second one, attributed to Green, defines a perfectly elastic body as one in which *the internal energy (or a related quantity) is a function of the strain*[14]. Although any reference whatsoever to atomic approach is purposely extraneous to the interests of the American scholar[15], this distinction may be perfectly transferred to the original formulation of elasticity in molecular terms: and more than that, it would perhaps be correct to sustain, at least from a historical point of view, that this is better applied with respect to such a formulation.

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[11] G. WERTHEIM, *Mémoire sur l'équilibre des corps solides homogènes*, Annales de Chimie et de Physique, III sér., vol. 23, 1848, pp. 52-95.

[12] R. CLAUSIUS, *Ueber die Veränderungen, welche in den bisher gebräulichen Formeln für das Gleichgewicht und die Bewegung elasticher fester Körper durch neuere Beobachtungen nothwendig geworden sind*, Annalen der Physik und Chemie, vol. 76, 1849, pp. 46-67.

[13] After the initial adhesion to the molecular formulation in the *Mémoire sur l'équilibre intérieur des corps solides homogènes* (Mém. préséntes par divers savans, vol. 4, 1833, pp. 465-562) written in collaboration with Clapeyron, the famous *Leçons sur la théorie mathématique de l'élasticité des corps solides* (Paris, 1852) constitute a first and still uncertain change in direction with respect to Lamé's first line of research. His acceptance of the bi-constant formulae, justified on the basis of the experimental results by Wertheim, contradicts the constant reference to the molecular model.

[14] C. TRUESDELL, *Continuum Mechanics I. The Mechanical Foundations of Elasticity and Fluid Dynamics*, The international science review series, 8, New York-London-Paris, 1966, p. 52.

[15] Truesdell's line of thought is clearly expressed in *Appendix 1* of the *Mechanical Foundations*: *I hope that what little I can present (...) will encourage the reader to question the all too common assumption that because physical matter is composed of molecules, a theory based on the crudest and most unrealistic molecular hypothesis is automatically preferable to any continuum theory. Indeed, I contend that gross phenomena are most naturally, accurately, and elegantly represented by gross hypotheses alone* (p. 189).

In effect, when Green[16] laid the foundations of the theory of elasticity in the name of the *grand principle*, as Thomson and Tait[17] later called the principle of conservation of energy, the starting point was constituted by the criticisms to the *rather restrictive supposition* on the basis of which Cauchy had admitted the action between two molecules directed along their joining line and depending exclusively on their mutual distance. Therefore, not the *tout court* refusal of the molecular model in favour of the continuum hypothesis but, more simply, a "suspension of judgement" on the modality of action between the last elements of matter.

This attitude is not new in the development of scientific thought: it surprisingly reflects the epistemological position assumed almost one century before by Jacopo Riccati[18] against the alternative systems of Cartesio, Newton and Leibniz. Instead of attributing the statute of *nozioni di cose* to the "subtle matter" of the first, to the "attractions and repulsions at a distance" of the second, to the "monads" of the latter, thus giving these a real physical consinstency, the Italian scientist was inclined for inscribing them amid those *nozioni di metodo* useful in accounting for the "external properties" of matter, in other words, to describe material behaviour from a phenomenological point of view. Therefore it is not surprising that, in search of an universal mathematical law independent of any mental representation whatsoever of the constitution of matter, Riccati reconducted the "elastic virtue" of bodies to the perpetual and uninterrupted passage of living forces in dead forces and vice versa, that is to say, to the unquestionable principle of conservation of energy used as a reference with the same intention by Green.

The axiom assumed by the English author *as the basis of the reasoning* states that *in whatever way the elements of any material system may act upon each other, if all the internal forces exerted be multiplied by the elements of their respective directions, the total sum for any assigned portion of the mass will always be the exact differential of some function*[19]. The consequences deduced on this basis are well known: having admitted this certain function expressing the internal work of deformation and depending on the six components of strain ε_{ij}, that is $\Phi = F(\varepsilon_{ij})$; supposed these latter infinitesimal

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[16] G. GREEN, *On the Laws of Reflection and Refraction of Light at the common Surface of Two Non-crystallized Media*, Transactions of the Cambridge Philosophical Society, vol. 7, (1838-1842), 1839, pp. 1-24. Also in G.Green, *Mathematical and Physical Papers of the late George Green*, London, 1871, pp. 243-269; pp. 281-290.

[17] W. THOMSON, P. G. TAIT, *Treatise on Natural Philosophy*, I, Oxford, 1867, p. vi.

[18] J. RICCATI, *Saggio intorno il Sistema dell'Universo, Opere*, Lucca, 1761. Regarding the thought of Riccati see E. BENVENUTO, M. CORRADI, G.PIGAFETTA, *Contributi italiani alla scienza delle costruzioni*, in *La cultura filosofica e scientifica, II: La storia delle scienze*, 1989, pp. 875-938.

[19] G. GREEN, op. cit., p. 245.

of first order; developed the function Φ in a series $\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \Phi_3 + \ldots$ in the integer powers of the components ε_{ij}; assumed the initial state as the state of equilibrium, resulting in $\Phi_1 = 0$; and overlooked the terms of the series higher than the second one, the internal work is reduced, except the constant term Φ_0, to a quadratic function Φ_2 in the strain components and, as such, containing 21 distinct coefficients in the general case of aelotropy, which are reduced to 2 for isotropic bodies.

These conclusions, which are perfectly compatible with Cauchy's continuum theory, were accepted without reserve by the whole Anglo-saxon school, even though with distinct premises. Stokes[20] and Maxwell[21], for example, derived the origin of bi-constancy from the conviction that the material behaviour would be characterised by the contemporaneous presence of *elasticity of volume* and *elasticity of shape* in a variable ratio according to the different substances.

Rankine[22] interpreted this idea from an atomistic point of view by admitting that the *hypothesis of Boscovich*, that is the old molecular model, would account for the behaviour of the "perfect solids"; on the contrary, the "imperfect solids" would be connoted by an increase J in the *elasticity of volume* responsible for the failure of the Cauchy relation $k = K$. That's not all. Following the same line of thought, Jellett[23] suggested a complex classification of material behaviour based on two different hypotheses of molecular interaction: the former, called *hypothesis of independent action*, would be valid for those bodies *whose particles exert upon each other a force which is* independent *of the surrounding particles*, as implied in the treatments of French molecularists: in this case the potential function takes the particular form

$$\Phi = F_1(r_1) + F_2(r_2) + F_3(r_3) + \ldots$$

where r_1, r_2, r_3 etc. are the distances between the particles; the latter, defined as the *hypothesis of modified action*, would be pertinent to those bodies for which *the mutual action of two particles is supposed to be affected by that of the surrounding particles*: in this case the potential assumes the general expression

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[20] G. G. STOKES, *On the theories of internal friction of fluids in motion, and of the equilibrium and motion of elastic solids*, Trans. of the Cambridge Philosophical Society, vol. 8, 1847, pp. 287-319.

[21] J. C. MAXWELL, *On the Equilibrium of Elastic Solids*, Trans. of the Royal Society of Edinburgh, vol. 20, 1853, pp. 87-120.

[22] W. J. M. RANKINE, *Laws of Elasticity of Solid Bodies*, Cambridge and Dublin Mathematical Journal, vol. 6, 1851, pp. 47-80; pp. 178-181; pp. 185-186.

[23] J. H. JELLETT, *On the Equilibrium and Motion of an Elastic Solid*, Trans. of the Royal Society Academy, vol. 22, III, 1852, pp. 179-217.

$$\Phi = F(r_1, r_2, r_3, \ldots),$$

which is implicit in Green's methodological approach and in the rigorous thermodynamical foundation of elastic phenomena given by Thomson[24].

The controversy on the elastic constants regarding Saint-Venant's work

The defective character that we have recognized in Todhunter and Pearson's *History* is due to the persisting controversy that, on the wave of the diverging conclusions seen before and without resolutive experimental results, should have opposed, till the close of the XIX century, the supporters of the *multi-constant* theory against those of the *rari-constant* theory.

Amongst these latter, a role of absolute importance has to be ascribed to the *Ingénieur des ponts et chaussées* A.J.C. Barré de Saint-Venant. From the *Leçons de mécanique* [25] held at the École during the years 1837-'38, where the molecular definition of stress first given by Cauchy and Poisson is rigorously presented, to the *Mémoire sur la question de savoir s'il existe des masses continues (...)*[26], in which the profession of faith to the *système de Boscovich* becomes explicit; from the famous "technical" memoirs on the torsion and flexion of prisms[27], where the need emerges to inscribe the application problems within the general molecular formulation, to the *Appendices* added to the third edition of the *Résumé des Leçons*[28] by Navier, representing an acquired synthesis

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[24] W. THOMSON, *On the thermo-elastic and thermo-magnetic properties of matter*, Quarterly Journal of pure and applied mathematics, vol. 1, 1857, pp. 57-77.

[25] A. J. C. BARRÉ DE SAINT-VENANT, *Leçons de mécanique faites par intérim par M. de Saint-Venant, Ingénieur des ponts et chaussées, 1837 à 1838*, Paris.

[26] A. J. C. BARRÉ DE SAINT-VENANT, *Mémoire sur la question de savoir s'il existe des masses continues, et sur la nature probable des dernières particules des corps*, Société Philomatique de Paris, 1844, pp. 3-15; see also the memoir *De la constitution des atomes*, Annales de la Société scientifique de Bruxelles, vol. 2, 1877-'78, pp. 417-456, *Complement*, pp. 1-39).

[27] A.J.C.BARRÉ DE SAINT-VENANT, *Mémoire sur la torsion des prismes, avec des considérations sur leur flexion ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance a divers efforts s'exerçant simultanément*, Mémoires présentés par divers savants à l'Académie des Sciences de l'Institut Impérial de France, vol. 14, 1855, pp. 233-560; *Mémoire sur la flexion des prismes, sur les glissement transversaux et longitudinaux qui l'accompagnent lorqu'elle ne s'opèrs pas uniformément ou en arc de cercle, et sur la forme courbe affectée alors par leurs sections transversales primitivement planes*, Journal de mathématiques pures et appliquées, II sér., vol. 1, 1856, pp. 89-189.

[28] *Résumé des Leçons données à l'École des Ponts et Chaussées sur l'application de la Mécanique à l'établissement des constructions et des machines, par Navier, avec des Notes et des Appendices par M. Barré de Saint-Venant*, Paris, 1864^{III}.

between the practical requirements of the *Résistance des Matériaux* and the theoretical instances of the *mécanique moléculaire*: the entire scientific curriculum of the French scholar is marked by a constant reference to the atomistic hypothesis.

The reason for this obstinate position has a very important epistemological significance. There are grounds to believe that the main theoretical point of the dispute between molecular and continuum formulation resides in the concept itself of force or stress, that is to say, in the alternative between two positions of thought. The one which has become an integral part of the modern *continuum mechanics* and its mathematical axiomatisation, according to which - by quoting Truesdell - *forces and torques, like bodies, motions, and masses, are primitive elements of mechanics*[29], and the one deriving from the old program conceived by D'Alembert and Carnot in order to found mechanics on a strictly kinematical base, according to which the title "primitive element" of mechanics only concerns the concepts of body and motion, whilst force, and also stress, must be considered as derived quantities.

Following this second line of thought, since 1845 Saint-Venant[30] had re-interpreted in geometrical terms the law of dynamics of the material point

$$\frac{dv}{dt} = F_1 + F_2 + F_3 + \dots, \tag{1}$$

with the explicit aim to avoid any reference to the obscure idea of force like the efficient cause of motion; he saw as the first member a geometrical differential coefficient, *le* flux géométrique *de la vitesse v*, and as the second member the sum of the partial fluxes of velocity, each one representing the effective flux that the material point would take if each *circonstance de position* in which it finds itself with rispect to the surrounding points took place alone, without the other ones.

This epistemological perspective, made explicit in a didactic text on the *Principes de mécanique fondés sur la cinématique* (1851) whose role is directly connected to the debate on the principles of mechanics developed towards the end of the XIX century[31], was destined to re-emerge with all its consequences in the *Notes* to the French translation

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[29] C. TRUESDELL, *A First Course in Rational Continuum Mechanics*, New York-San Francisco-London, 1977, p. 119.

[30] A. J. C. BARRÉ DE SAINT-VENANT, *Mémoire sur les sommes et les différences géométriques, et sur leur usage pour simplifier la Mécanique*, Comptes rendus, vol. 21, II sem., 1845, pp. 620-625, in particular pp. 624-625.

[31] H. PADÉ, *Barré de Saint-Venant et les principes de la Mécanique*, Revue générale des sciences pures et appliquées, vol. 15, 1904, pp. 761-767.

of the *Theorie der Elasticität* by Clebsch[32]: in this circumstance Saint-Venant positively rejected the *plus générale parce que moins détérminée* hypothesis of molecular action implicit in Green's theory, since it would deny the geometrical clearness reflected in the law (1) of the kinematical composition of motions, imposing an ontological dimension to the concept of force in virtue of a purely mathematical principle. From this point of view, his conclusion could hardly be different: *j'affirme hardiment, et tout le monde ... pensera comme moi*, concluded the old scientist, *qu'il faudra absolument adopter la forme* [33]

$$\Phi = F_1(r_1) + F_2(r_2) + F_3(r_3) + \dots \quad .$$

As known, history proved that Saint-Venant was wrong: not only has the sistematic choice of principles of mathematical nature permitted major progresses in present rational mechanics, but the achievements of modern solid state physics have shown[34] that the material behaviour can be formally interpreted only by admitting that cohesive forces originate from many-body potentials, exactly as intended by Jellett with his second hypothesis.

Even if Saint-Venant's theoretical work is highly compromised by the atomistic prejudice[35], it must be remembered, however, that, in front of the untiring denial of the multi-constant formulae, he did not confine himself to a barren opposition of the rari-constant ones. Behind the rejection of the laboratory results contrasting with the original molecular theory, one discloses an unremitting effort in the research of plausible physical reasons to the experimental failure of the uni-coefficient relations. We want to refer here to a conspicuous number of studies[36] that the French scientist developed with the aim to

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[32] A. CLEBSCH, *Théorie de l'élasticité des corps solides, traduite par MM. Barré de Saint-Venant et Flamant, avec des Notes étendues de M. de Saint-Venant*, Paris, 1883.

[33] *Ibidem*, p. 72.

[34] P. O. LÖWDIN, *A theoretical investigation into some properties of ionic crystals. A quantum mechanical treatment of the cohesive energy, the interionic distance, the elastic constants, and the compression at high pressures with numerical applications to some alkali halides*, Uppsala, 1948; *A quantum mechanical calculation of the cohesive energy, the interionic distance, and the elastic constants of some ionic crystals*, Arkiv för Matematik, Astronomi och Fysik, vol. 35 A, 9, 1948, pp. 1-10; II, *The elastic constants c_{12} and c_{44}*, Idem, 30, pp. 1-18.

[35] E. BENVENUTO, A. BECCHI, *Sui principi di filosofia naturale che orientarono la ricerca di Saint-Venant*, in *Omaggio a Giulio Ceradini*, Roma, 1988, pp. 125-133.

[36] A. J. C. BARRÉ DE SAINT-VENANT, *Mémoire sur la distribution des élasticités autour de chaque point d'un solide ou d'un milieu de contexture quelconque, particulièrement lorsqu'il est amorphe sans être isotrope*, Journal de mathématiques pures et appliquées, II sér., vol. 8, 1863, pp. 257-295; pp. 353-430; *Mémoire sur les divers genres d'homogénéité des corps solides, et principalement sur l'homogénéité semi-polaire ou cylindrique, et sur les homogénéités polaires ou sphériconique et spherique*, Journal de mathématiques pures et appliquées, II sér., vol. 10, 1865, pp. 297-350; *Formules de l'élasticité des corps amorphes que des compressions permanentes et inégales ont rendus hétérotropes*, Journal de

re-interpret the tests by Wertheim and Kirchhoff[37] using the new idea of "amorphous body", introduced for the first time in 1863 in order to describe the mechanical properties acquired by initially isotropic solids after processing or geological formation. For these bodies, including metallic and stone materials, Saint-Venant suggested the adoption of equations with three distinct coefficients, definitely rejecting the *formules fautives d'isotropie à deux paramètres* since they had no physical reference to the *grand loi* of molecular action.

The last developments of the XIX century debate

This coherent line of research should not have found direct heirs. In the dismissal *Notice* to the deceased master, Boussinesq humbly took his distance from it observing that *il faudrait supposer .... que notre faculté de représentation du monde extérieure s'accorde assez bien avec la realité physique pour que sa compétence doive être admise sans restriction, jusque dans l'étude des derniers détails des objets*[38]. With this the ex-disciple reaffirmed the position already assumed in his *Recherches ... sur la constitution moléculaire des corps*[39], where he had implicitly agreed with the Anglo-saxon school of thought by admitting that the action between two material points may be affected by the presence of others.

Even still before Boussinesq, another blow to the molecular tradition was given by Lamé himself. Following a line of thought rather like the one that had brought Franz Neumann[40] to revise his initial adhesion to the rari-constant theory, the eminent French scientist had formulated his *Leçons sur les coordonnées curvilignes* (1859) on the definite rejection of the *ancien principe* of the molecular theory, thus explicitly inauguring that "mathematical" approach to elasticity anticipated in his *Leçons* of 1852 and then followed

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mathématiques pures et appliquées, II sér., vol. 13, 1868, pp. 242-254; *Des paramètres d'élasticité des solides, et de leur determination expérimentale*, Comptes rendus, vol. 86, I sem., 1878, pp. 781-785.

[37] G. KIRCHHOFF, *Ueber das Verhältniß der Quercontraction zur Längendilatation bei Stäben von federhartem Stahl*, Annalen der Physik und Chemie, vol. 108, 1859, pp. 369-392.

[38] J. BOUSSINESQ, A. A. FLAMANT, *Notice sur la vie et les travaux de Barré de Saint-Venant*, Paris, 1886, p. 21.

[39] J. BOUSSINESQ, *Recherches sur les principes de la Mécanique, sur la constitution moléculaire des corps et sur une nouvelle théorie des gas parfaits*, Journal de mathématiques pures et appliquées, vol.18, 1873, pp. 305-360.

[40] Regarding the evolution of Neumann's ideas see the *Vorlesungen über die Theorie der Elasticität der festen Körper und des Lichtäthers* (Leipzig, 1885) collected by O. E. Meyer and the *Gesammelte Werke* (in particular vol. II, 1906) edited by the son Carl.

in nowadays classical texts, from Clebsch's *Theorie* to Kirchhoff's *Vorlesungen* up to Love's *Treatise*.

Nevertheless, even thirty years after the net taking of stand by Lamé, the price of his abstract formulation should have appeared too high for those who did not want to renounce a physical foundation of the phenomenological datum. It was actually Pearson who took up position against his approach to elasticity, denouncing it as *an apparent miracle (...) springing created from the brain of a mathematician without any appeal to experience*[41]. It is known, on the other hand, the substantial disagreement of the Anglo-saxon scholar towards the thesis of the multi-constant compatriots, so that Green's method should only be *a chain of arbitrary assumptions* if deprived of the physical reference to molecular actions, whilst the suggestive idea by Stokes and Maxwell could only be conceivable in a non linear theory; and, if Jellett's hypothesis of interaction *of a polar character* cannot be rejected *a priori*, it would be difficult to accept it for isotropic bodies for which, on the other hand, it may not be said that the experiments *have absolutely settled the controversy in favour of multi-constancy*.

It would be so far too easy to liquidate these criticisms by reducing them to a mere debt of recognition towards the figure of Saint-Venant. The unconditioned appreciation of the *great French elastician* expressed by Pearson right from the dedication in the first volume of the *History* reflects in reality a conviction still diffused at the end of the XIX century, resulting from the persistent reference to the first molecular model. The image on which Bravais[42] had formed his studies on the *systèmes formés par des points distribués régulièrement* translated in an eloquent manner the more general idea of crystalline solid. The simple addition of elastic rods to connect those points allowed Kirsch[43] to obtain a brilliant deduction of the equations of elasticity in accordance with the rari-constant theory. Not only. This same model had been used few years earlier by Menabrea[44] to lay the foundations of the general methods of structural mechanics; not because the concept of energy of deformation placed as the basis of his "principle of elasticity" required a foundation in terms of molecular work, but because the reduction of the "elastic system" to an assembly of hinges connected by deformable rods was able to offer the universal image of the solid body as the object of study of the general theory of elasticity, so as to

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[41] I. TODHUNTER, K. PEARSON, op. cit., I, p. 625.

[42] A. BRAVAIS, *Mémoire sur les systèmes formés par des points distribués régulièrement sur un plan ou dans l'éspace*, Journal de l'École polytechnique, vol. 19, Cahier 33, 1850, pp. 1-128.

[43] G. E. KIRSCH, *Fundamentalgleichungen der Theorie der Elasticität fester Körper, hergeleitet aus der Betrachtung eines Systems von Punkten, welche durch elastische Streben verbunden sind*, Zeitschrift des Vereines deutscher Ingenieure, vol. 12, 1868, pp. 481-481; pp. 553-570; pp. 631-638.

[44] L.F.MENABREA, *Nouveau principe sur la distribution destensions dans les systèmes élastiques*, Comptes rendus, vol. 46, 1858, pp. 1056-1060.

motivate the doubts brought about by Cerruti[45] concerning the possibility of extending that principle from the case of a discrete number of point connected by elastic wires to the case of a continuum body.

Therefore it is not surprising that the binomial *Elastizität und Festigkeit*, intended to signify the close relationship between the theoretical premises of mechanics of solids and the applications of mechanics of structures during the second half of the XIX century, often hides a more or less explicit adhesion to the rari-constant theory[46]. This is so in the treatises by Beer[47] and Grashof[48], loyally submitted to the original French formulation, and in Weyrauch's text[49], although the treatment in molecular terms ambiguously cohabits with the continuum one; this is also the case in Castigliano's *Théorie*[50] or in the later *Lezioni sulla Scienza delle costruzioni* by Guidi[51].

Paris 1900: towards the epilogue of the *History*

With this last quotation we have arrived at the end of our narration, having completely crossed the chronological threshold of the *History* by Todhunter and Pearson. The unchanged draft of the chapter on the "General theory of elasticity" by which the *Lezioni* were reissued several times during the first decades of the XX century makes Guidi's text hopelessly obsolete, especially when compared with the almost contemporary work by Marcolongo[52], which is perfectly up-to-date with the last achievements of the theoretical-experimental research. After decades of contradictory investigations, made vain by the persistent uncertainty of the real isotropy of the tested

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[45] V. CERRUTI, *Sopra un teorema del Sig. Menabrea*, Atti della Regia Accademia dei Lincei, vol.2, ser. II, 1875, pp. 570-581.

[46] Regarding this point we refer to F. FOCE, *Die Auswirkungen des Streits zwischen Molekular- und Kontinuumshypothese in der Elastizitätstheorie auf die Entwicklung der Strukturmechanik*, Report held at the "Jahrestagung der Gesellschaft für Wissenschafts- und Technikforschung am Wissenschaftszentrum Berlin für Sozialforschung", Berlin, December 2-4, 1993.

[47] A. BEER, *Einleitung in die mathematische Theorie der Elasticität und Capillarität*, Leipzig, 1869.

[48] F. GRASHOF, *Theorie der Elasticität und Festigkeit mit Bezug auf ihre Anwendungen in der Technik*, Berlin, 1878.

[49] J.J. WEYRAUCH, *Theorie elastischer Körper. Eine Einleitung zur mathematischen Physik und technischen Mechanik*, Leipzig, 1884.

[50] A. CASTIGLIANO, *Théorie de l'équilibre des systèmes élastiques et ses applications*, Turin, 1879.

[51] C. GUIDI, *Lezioni sulla Scienza delle costruzioni, II. Teoria dell'elasticità e resistenza dei materiali*, Torino, 1909^{V}.

[52] R. MARCOLONGO, *Teoria matematica dell'equilibrio dei corpi elastici*, Milano, 1904.

materials[53], the fundamental studies led by Voigt[54] on the elasticity of aelotropic bodies had unquestionable shown the experimental failure of the Cauchy relations, pointing out the need to re-formulate the old molecular theory within the framework of a renewed theory of action at a distance[55]. Interpreting the modern methodological approach assumed by Poincaré[56] in front of the two "indifferent" hypotheses[57] on the constitution of matter, the famous crystallographer from Göttingen had finally to overcome the dispute between molecular and continuum formulation, recognizing in their mere "formal" agreement the only necessary requisite for results coming from opposite representations of the material universe.

From this point of view, the year 1900 signs the end of the debate started 67 years before on the *Bulletin de la Société philomatique*. The two International Congresses of Mathematics and Physics which took place in Paris at that date certify the start of a new era, connoted by the definite attribution of competence between mathematicians and phycisists that has characterized the development of science during the present century. The VIth of the *Mathematische Probleme* proposed by Hilbert[58] to the future generations was destined to be solved with the axiomatic approach of the present *continuum*

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[53] After an initial confirmation of the uni-constant formulae through the experiments by C.CAGNIARD DE LA TOUR (*Note sur l'élasticité des cordes métalliques*, La Globe, vol. 6, n° 19, 1828, pp. 107-108) and their first confutation due to the mentioned tests by Wertheim and Kirchhoff, new reassurances of the result of the molecular theory were obtained with the optical measurements led by A. CORNU on beams of glass (*Méthode optique pour l'étude de la déformation de la surface extérieure des solides élastiques*, Comptes rendus, vol. 69, II sem., 1869, pp. 333-337), in their turn disproved by W.VOIGT's tests on the same material (*Ueber das Verhältnis der Quercontraction zur Längendilatation bei Stäben von isotropem Glas*, Annalen der Physik und Chemie, vol. 15, 1882, pp. 497-513). Further results in favour of the uni-constant formulae were obtained through the piezometer experiments by M. CANTONE (*Nuovo metodo per la determinazione delle due costanti di elasticità*, Rendiconti della Reale Accademia dei Lincei, IV ser., vol. 4, 1888, pp. 220-227; pp. 292-297) and by É.H.AMAGAT (*Sur la vérification expérimentale des formules de Lamé et la valeur de coefficient de Poisson*, Comptes rendus, vol. 106, 1888, pp. 479-482; *Recherches sur l'élasticité des solides*, Comptes rendus, vol. 108, 1889, pp. 1199-1202). For more detailes on the experimental research see J.F. BELL, *The experimental foundations of solid mechanics*, Handbuch der Physik, 6 a/I, *Festkörpermechanik I*, Berlin-Heidelberg-New York, 1973.

[54] W. VOIGT, *Bestimmung der Elasticitätsconstanten von Beryll und Bergkrystall*, Annalen der Physik und Chemie, vol. 31, 1887, pp. 474-501; pp. 701-724; *Bestimmung der Elasicitätsconstanten von Topas und Baryt*, idem, vol. 34, 1888, pp. 981-1028; *Bestimmung der Elasicitätsconstanten von Flussspath, Pyrit, Steinsalz, Sylvin*, Idem, vol. 35, 1888, pp. 642-661.

[55] W. VOIGT, *Theoretische Studien über die Elasticitätsverhältnisse der Krystalle*, Abhandlungen der königlichen Gesellschaft der Wissenschaften zu Göttingen, vol. 34, 1887, pp. 3-100; in particular, part I, *Ableitung der Grundgleichungen aus der Annahme mit Polarität begabter Moleküle*. See also *Lehrbuch der Kristallphysik*, Leipzig, 1928^{II}, pp. 596-616.

[56] H. POINCARÉ, *Leçons sur la théorie mathématique de la lumière*, Paris, 1889; *Leçons sur la théorie de l'élasticité, redigées par MM. Emil Borel et Jules Drach*, Paris, 1892.

[57] H. POINCARÉ, *Science and hypothesis*, New York, 1952.

[58] D. HILBERT, *Mathematische Probleme*, Archiv der Mathematik und Physik, 1, III Folge., 1901, pp. 44-63; pp. 213-237; also in French with the title *Sur les problèmes futurs des mathématiques*, Compte rendu du deuxième Congrès International des Mathématiciens, Paris, 1902, pp. 59-114.

mechanics pursued by Truesdell's school in line with the first Cauchy[59], whilst the new theory of action at a distance (*Fernwirkungstheorie)* by Voigt[60] should have updated the instances of the old *mécanique moléculaire* in the framework of the modern *Theory of crystal lattices* systematized shortly afterwards by Born[61], thus conserving the ambition to ensure a "per causas" foundation of the elasticity of solids in the same way in which the alternative theory of contact action (*Nahwirkungstheorie*) was able to offer a "mathematical representation" of it.

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[59] C. TRUESDELL, *Cauchy and the modern mechanics of continua*, Revue d'histoire des sciences, vol. 45, 1992, pp. 5-24.

[60] W. VOIGT, *Die gegenwärtige Stand unserer Kenntnisse der Krystallelasticität, Referat für den internationalen physikalischen Congreß in Paris vom 6. bis 12. August 1900*, Nachrichten von der königlichen Gesellschaft der Wissenschaften zu Göttingen, 1900, pp. 117-176; also in French with the title *L'état actuel de nos connaissances sur l'élasticité des cristaux*, Rapport présenté au Congrès international de Physique reuni à Paris en 1900, Paris, 1900, pp. 1-71.

[61] M. BORN, *Dynamik der Kristallgitter*, Leipzig und Berlin, 1915. *Atomtheorie des festen Zustandes*, Encyclopädie der mathematischen Wissenschaften, vol. 5, III, 1909-1926, pp. 527-781.